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Multifractal statistics of multiparticle production at high energies

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Abstract. It is shown that high energy multiparticle production near the morphological phase transition from monofractality to multifractality is characterized by a multifractal Bernoulli distribution. Experimental data on hadron-hadron, hadron-nucleus and on heavy ions collisions are used to show an universal character of this distribution (and, consequently, of the morphological phase transition) in multiparticle production at high energies.

PACS. 13.85.Hd Inelastic scattering: many particle final state – 24.60.Ky Fluctuation phenomena

1 Introduction

Statistics of multiparticle production at high energies are studied very intensively in the last decade (see for recent reviews [1,2]). There are two ways to describe the statistics: 1) using probability density or characteristic functions, 2) using a system of moments. In papers [3] the second way was successfully applied to interpret some experimental data on multiparticle production at high energies. Suppose that underlying dynamics is determined by a cascade process the authors of [3] suggested log-normal distribution to describe the experimental data represented in a form of a system of factorial moments. Another direction of the investigation implies the phase-transition like interpretation [4], [5]. Monofractal picture (from which this approach starts [4]) leads to simple Bernoulli distribution [6]. Then more complex cascade and phase-transition like models were suggested and correspondingly more complex known probability distributions (such as Levy stable law and negative binomial distribution) were used.

It is clear that multifractal dynamics appears as a development of the monofractal states (this development can take a critical character). On the other hand, we don't know how this transition proceeds and what probability law should be used for its description. It seems naturally that this probability law should play a significant role in the multifractal processes, due to its presumably universal (as it usually takes place at critical phenomena) character. Especially, if the multiparticle production processes occur near the transition. Therefore, in the present paper, we will find and describe in an explicit form a probability law corresponding to the transition from monofractality to multifractality (multifractal Bernoulli distribution). Then we check whether the experimental data on different high energy production processes can be fitted by this transitional distribution. It was a surprise that experimental data on hadron-hadron, hadron-nucleus and on very heavy ions collisions at high energies not only are in good agreement with the multifractal Bernoulli distribution, but also some (thermodynamic) parameter of this distribution (so-called multifractal specific heat) takes approximately the same constant value ($\simeq 1/4$) in all these processes (cf. also a preliminary communication [7]). So that universality of this transitional distribution seems to be an indication of critical character of all these reactions. On the other hand, a constant value of the multifractal specific heat implies that we are far away from the 'transitional point' itself. This unusual property of the multifractal thermodynamics of the multiparticle production can be related to the unusual thermodynamics properties of dynamical systems at the onset of chaos and we consider this question elsewhere [7].

2 Phase transition from monofractality to multifractality

Let $\Delta\eta$ be the pseudo-rapidity interval, and subdivide into M bins each of width $\delta\eta = \Delta\eta/M$. Let N be the number of particles in one event in $\Delta\eta$ interval and k_m be the number of particles in the m-th bin. The G_q moments are defined as [8]

$$G_q = \sum_{m=1}^{M} \mu_m^q \tag{1}$$

where $\mu_m = k_m/N$ is the probability of particles in the m-th bin for one event and q is any real number. The summation is carried out over non-empty bins only. If the multiparticle production process exhibit self-similar behavior then the moment follow the power law

$$G_q \propto (\delta\eta/\Delta\eta)^{\tau(q)}. \quad (2)$$

The generalized dimension spectrum is then given by:

$$D_q = \tau(q)/(q-1). \quad (3)$$

Then, if one uses standard averaging one obtains

$$\langle\mu^q\rangle = \frac{\sum_{i=1}^{M}[\mu_i(l)]^q}{M} \propto (\delta\eta/\Delta\eta)^{(\tau_q+1)} \quad (4)$$

Let us define

$$\overline{\mu_i} = \mu_i/\max_i\{\mu_i\}. \quad (5)$$

Then

$$\langle\overline{\mu}^p\rangle = \frac{1}{M}\sum_i \overline{\mu_i}^p. \quad (6)$$

The simplest structure, that can be used for fractal description, is a system for which $\overline{\mu_i}$ can take only two values 0 and 1. It follows from (5) that for such system (with $p > 0$)

$$\langle\overline{\mu}^p\rangle = \langle\overline{\mu}\rangle \quad (7)$$

and fluctuations in this system can be identified as Bernoulli fluctuations [6]. It is clear that the Bernoulli fluctuations can be *monofractal* only.

Generalization of (6) in form of a generalized scaling

$$\langle\overline{\mu}^p\rangle \sim \langle\overline{\mu}\rangle^{g(p)} \quad (8)$$

can be used to describe more complex (multifractal) systems. We use invariance of the generalized scaling (7) with dimension transform [9]

$$\overline{\mu_i} \to \overline{\mu_i}^{\lambda} \quad (9)$$

to find $g(p)$. This invariance means that

$$\langle(\overline{\mu}^{\lambda})^p\rangle \sim \langle(\overline{\mu}^{\lambda})\rangle^{g(p)} \quad (10)$$

for all positive λ. Then, it follows from (7) and (9) that

$$\langle(\overline{\mu})^{\lambda p}\rangle \sim \langle\overline{\mu}\rangle^{g(\lambda p)} \sim \langle\overline{\mu}\rangle^{g(\lambda)g(p)} \quad (11)$$

Hence,

$$g(\lambda p) = g(\lambda)g(p). \quad (12)$$

The general solution of the functional equation (11) is given by

$$g(p) = p^{\gamma} \quad (13)$$

where γ is a positive number. It should be noted that case $\gamma = 1$ corresponds to Gauss fluctuations [10]. We, however, shall consider the limit $\gamma \to 0$ (i.e. transition to the Bernoulli fluctuations). This transition is non-trivial. Indeed, let us consider generalized scaling

$$F_{qm} \sim F_{km}^{\alpha(q,k,m)} \quad (14)$$

where

$$F_{qm} = \langle\overline{\mu}^q\rangle/\langle\overline{\mu}^m\rangle \quad (15)$$

Substituting (7) into (13),(14) and using (12) we obtain

$$\alpha(q,k,m) = \frac{q^{\gamma} - m^{\gamma}}{k^{\gamma} - m^{\gamma}}.$$

Hence,

$$\lim_{\gamma\to 0}\alpha(q,k,m) = \frac{\ln(q/m)}{\ln(k/m)}. \quad (16)$$

If there is ordinary scaling

$$\langle\overline{\mu}^p\rangle \sim (\delta\eta/\Delta\eta)^{\zeta_p}, \quad (17)$$

then

$$\alpha(q,k,m) = \frac{\zeta_q - \zeta_m}{\zeta_k - \zeta_m} \quad (18)$$

Comparing (15) and (17) we obtain at the limit $\gamma \to 0$

$$\frac{\zeta_q - \zeta_m}{\zeta_k - \zeta_m} = \frac{\ln(q/m)}{\ln(k/m)}. \quad (19)$$

The general solution of the functional equation (18) is

$$\zeta_q = a + c\ln q, \quad (20)$$

where a and c are some constants.

If we use the relationship

$$\max_i\{\mu_i\} \sim (\delta\eta/\Delta\eta)^{D_\infty} \quad (21)$$

(see, for instance, [11]), then it follows from (3)–(5) and (16), (19), (20) that

$$D_q = D_\infty + c\frac{\ln q}{(q-1)} \quad (22)$$

for the multifractal Bernoulli fluctuations (i.e. for the fluctuations which appear at the limit $\gamma \to 0$).

3 Characteristic function

From (7), (16) and (19) we can find $g(p)$ corresponding to the multifractal Bernoulli fluctuations

$$g(p) = 1 + \frac{c}{a}\ln p \quad (23)$$

where $a = d - D_\infty$. One can see that for finite c the dimension-invariance is broken at the limit $\gamma \to 0$.

Let us find the characteristic function of the multifractal Bernoulli distribution. It is known that the characteristic function $\chi(\lambda)$ can be represented by following series (see, for instance [6])

$$\chi(\lambda) = \sum_{p=0}^{\infty}\frac{(i\lambda)^p}{p!}\langle\overline{\mu}^p\rangle \quad (24)$$

Then using (7) and (22) we obtain from (23)

$$\chi(\lambda) = 1 + \langle \overline{\mu} \rangle \sum_{p=1}^{\infty} \frac{(i\lambda)^p}{p!} p^{\beta} \quad (25)$$

where

$$\beta = \frac{c}{(d - D_{\infty})} \ln \langle \overline{\mu} \rangle \quad (26)$$

The characteristic function (24) gives complete description of the multifractal Bernoulli distribution. When $c = 0$ distribution (24)-(25) coincides with the simple Bernoulli distribution [6]. The multifractality-monofractality phase transition (with $\gamma \to 0$) corresponds to a gap from $c = 0$ to a finite non-zero value of c. If we use a thermodynamic interpretation of multifractality (see for a recent review [1]), then the constant c can be interpreted as multifractal specific heat of the system. The gap of the multifractal specific heat at the multifractality-monofractality transition (i.e. with $\gamma \to 0$) allows us consider this transition as a thermodynamic phase transition [1], [4] [5], [12].

4 Experimental data

Let us compare these theoretical results with laboratory data. Figure 1 (adapted from [13]) shows a generalized dimension spectrum D_q against variable $\ln(q)/(q-1)$. This generalized dimension spectrum (dots) was calculated in [13] using experimental data reported by the NA22 collaboration [14] who investigated the $\pi^+ p$ interactions with the centre-of-mass energy $(s)^{1/2} = 22GeV$. Calculations were performed in pseudo-rapidity phase space. Straight line in this figure indicates agreement of the data with the multifractal Bernoulli representation (21). If we calculate the multifractal specific heat from Fig. 1 we obtain $c \simeq 0.26 \pm 0.03$

In paper [15] results of intermittency in multiparticle production in proton interactions with various target nuclei in emulsion at 800 GeV (at Fermilab) are reported.

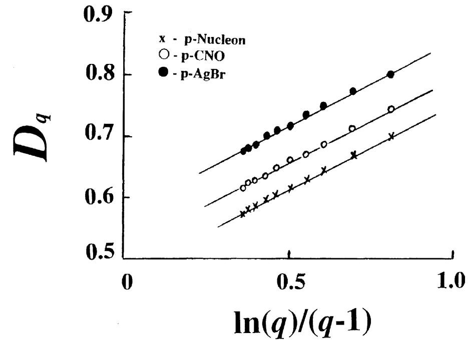

Fig. 2. Generalized dimension spectra (adapted from [15]) for multiparticle production in proton interactions with various target nuclei in emulsion at 800 GeV (at Fermilab). The straight lines are drawn for comparison with the multifractal Bernoulli representation (21)

Figure 2 (adapted from [15]) shows results obtained for p-Nucleon, p-CNO, and p-AgBr interactions. Calculations were performed in the pseudo-rapidity space. The axes in this figure are chosen so that straight lines indicate agreement of the data with the multifractal Bernoulli representation (21). The multifractal specific heat calculated from this figure – $c \simeq 0.27 \pm 0.01$.

The experimental spectrum (dots) shown in Fig. 3 was calculated in a recent paper [16] using the pseudo-rapidity phase space for the shower particles produced in the interactions of ^{197}Au emulsion at 10.6A GeV. The straight line in this figure indicates agreement between the data and the multifractal Bernoulli representation (21) with multifractal specific heat $c \simeq 0.23 \pm 0.02$. Analogous data on ^{28}Si ions collisions (also represented in [16]) don't give such clear indication of the morphological phase transition. This trend is confirmed by the data represented in paper [17] and obtained for projectile fragments in nuclear collisions at (1-2)A GeV. Figure 4 shows generalized dimension spectra calculated in [17] for ^{238}U at 0.96A GeV both in the pseudo-rapidity (lower set of dots) and in the

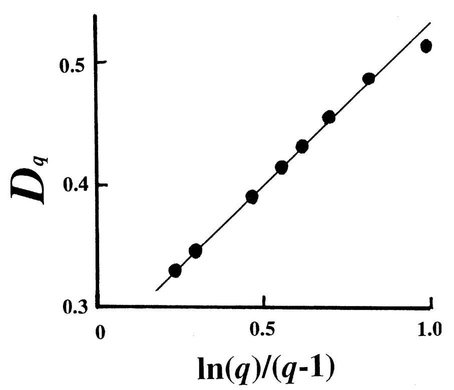

Fig. 1. Generalized dimension spectrum (adapted from [13]) for $\pi^+ p$ interactions with the centre-of-mass energy $(s)^{1/2} = 22GeV$. The straight line is drawn for comparison with the multifractal Bernoulli representation (21)

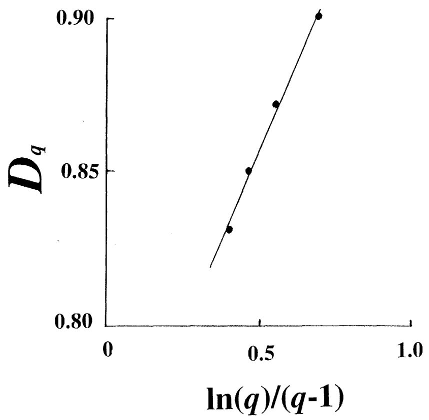

Fig. 3. Generalized dimension spectrum for ^{197}Au collisions on 10.6A GeV (dots). Data taken from [16]. The straight line is drawn for comparison with the multifractal Bernoulli representation (21)

Table 1. Values of multifractal specific heat, c, calculated using data shown in Figs. 1–4. "Hadron-hadron" column corresponds to π^+p interaction (Fig. 1), "hadron-nucleus" column corresponds to proton interactions with various target nuclei in emulsion (Fig. 2), "heavy ions 1" column corresponds to ^{197}Au collisions (Fig. 3). All these data are calculated in pseudo-rapidity phase space. "Heavy ions 2" and "heavy ions 3" columns correspond to ^{238}U collisions for data calculated in azimuthal and pseudo-rapidity phase spaces correspondingly (Fig. 4)

	hadron-hadron	hadron-nucleus	heavy ions 1	heavy ions 2	heavy ions3
C	$0.26 \div 0.03$	$0.27 \div 0.01$	$0.23 \div 0.02$	$0.26 \div 0.02$	$0.24 \div 0.02$

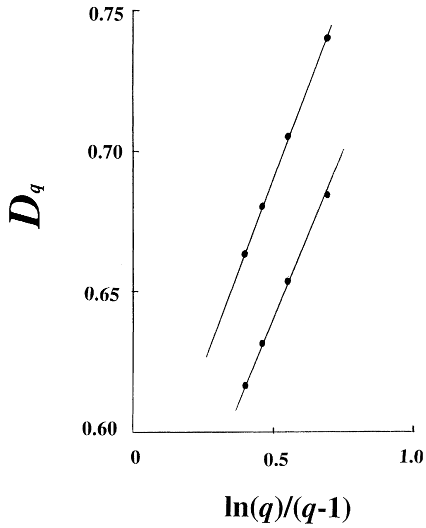

Fig. 4. Generalized dimension spectra for ^{238}U collisions at 0.96A GeV (dots). Data taken from [17]. Lower set of dots correspond to pseudo-rapidity phase space and upper set of dots corresponds to azimuthal phase space. The straight lines are drawn for comparison with the multifractal Bernoulli representation (21)

azimuthal (upper set of dots) phase spaces. And again the straight lines drawn in this figure indicate agreement between the data and the multifractal Bernoulli representation (21) with multifractal specific heat $c \simeq 0.26 \pm 0.02$ in the azimuthal space and $c \simeq 0.24 \pm 0.02$ in the pseudo-rapidity space. Analogous data calculated in paper [17] for ^{84}Kr and for ^{56}Fe ion collisions don't give such clear indication.

Thus, we can conclude that:

1. The experimental data presented here suggest a dominant role of the multifractal Bernoulli fluctuations in the multiparticle production at considered reactions (Figs. 1–4);

2. In all these reactions the multifractal specific heat is close to value 1/4 (Table 1).

It should be also noted that considered here reactions correspond to very different energy depositions. Therefore the question is: Should the *morphological* (monofractality-multifractality) phase transition be related to a *dynamical* phase transition (in present context – to the liquid-gas phase transition, because all reactions presented here are expected to lie below the transition to the quark-gluon phase [18]). If one gives a positive answer to this question, then the next questions are: a) What experimental parameter controls the monofractality-multifractality phase transition, and b) Which reactions should be investigated to observe the monofractal case with $c = 0$ (cf. [4])? In present stage we cannot give definite answers to these important questions and we hope that these problems will stimulate future investigations (both theoretical and experimental).

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